

ABSENCE OF SELF-SIMILAR BLOW-UP AND LOCAL WELL-POSEDNESS FOR THE CONSTANT MEAN-CURVATURE WAVE EQUATION

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ABSTRACT. In this note, we consider the constant-mean-curvature wave equation in $(1 + 2)$ -dimensions. We show that it does not admit any self-similar blow-up. We also remark that the equation is locally well-posed for initial data in $\dot{H}^{3/2}$.

1. INTRODUCTION

The study of nonlinear wave equations has attracted a lot of interest in recent years. One equation of interest is called the wave map. Recall that a wave map from the Minkowski space $\mathbb{R}^n \times \mathbb{R}$ into a m -dimensional Riemannian manifold M is a map $u: \mathbb{R}^n \times \mathbb{R} \rightarrow M$, which in local coordinates satisfies

$$(1) \quad (-\partial_t^2 + \Delta)u^a = -\Gamma_{bc}^a(u)(\partial^\alpha u^b)(\partial_\alpha u^c).$$

Here $\{u^a\}_{a=1}^m$ are the local coordinates of the point $u(x, t)$ in M , Γ_{bc}^a are the Christoffel symbols of M in the corresponding local coordinates, and we use Einstein's summation notation, so that repeated indices on the right hand side are summed (the sum of α is over all $\alpha = 0, 1, \dots, n$; we point out that the indices α are raised using the Minkowski metric on $\mathbb{R}^n \times \mathbb{R}$). Thus wave maps are just wave analogs of harmonic maps, which has been intensely studied for their connections to geometry and physics.

The equation (1) is critical for initial data in $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$. Well-posedness for initial data slightly above, or at critical regularity, has been studied by many authors; we recall some of these shortly. One important aspect that arise in the analysis of wave maps is the *null structure* of the non-linearity of the wave map equations. This has long been recognized, since the pioneering work of Klainerman and Machedon [9]. The simplest prototypes of null forms can be displayed

as follows:

$$\begin{aligned}
 Q_{00}(u, v) &= -(\partial_t u)(\partial_t v) + \nabla_x u \cdot \nabla_x v \\
 (2) \quad Q_{ij}(u, v) &= (\partial_{x_i} u)(\partial_{x_j} v) - (\partial_{x_j} u)(\partial_{x_i} v), \quad i, j \in \{1, \dots, n\} \\
 Q_{0j}(u, v) &= (\partial_t u)(\partial_{x_j} v) - (\partial_{x_j} u)(\partial_t v), \quad i \in \{1, \dots, n\}.
 \end{aligned}$$

Null forms of type Q_{00} arise in the study of wave maps. Using this null structure and the wave Sobolev $X^{s,b}$ spaces, Klainerman-Machedon [10], [11] and Klainerman-Selberg [12], [13] established subcritical local well-posedness for initial data in $H^s \times H^{s-1}$, $s > n/2$. At the critical regularity, one needs to bring in more geometric structures of the wave map equations, and write the equation in appropriate gauges. Tataru [30] proved that if the target M is uniformly isometrically embedded into some Euclidean space, then one has global existence for smooth initial data that has small $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$ norm, with control of the $L_t^\infty \dot{H}_x^{n/2}$ norm of the solution. The case where the target M is a sphere was obtained earlier in Tao [26], [27] by introducing what is called a microlocal gauge, and the case where $n = 2$ and $M =$ the hyperbolic plane was also in Krieger [15]. Krieger-Schlag [16] later extended this latter result to the case of large initial data. Furthermore, Sterbenz-Tataru [21], [22] proved that one has global existence and regularity for wave maps from $\mathbb{R}^n \times \mathbb{R}$ into M if the energy of initial data is smaller than the energy of any nontrivial harmonic map $\mathbb{R}^n \rightarrow M$.

On the other hand, Krieger-Schlag-Tataru [17] proved the existence of equivariant finite time blow-up solutions for the wave map problem from \mathbb{R}^{1+2} to \mathbb{S}^2 . Later, Rodnianski-Sterbenz [19] and Raphael-Rodnianski [18] considered corotational wave maps from 2 space dimensions into the sphere \mathbb{S}^2 with initial data in $\dot{H}^1 \times L^2$, and exhibited an open subset of initial data in any given homotopy class that leads to finite time blow up. In fact they have also obtained a rather precise blow-up rate of the blow-ups they constructed.

In this paper, we study another system of wave equations, with a different null-structure. It is the wave analog of the equation that prescribes constant mean curvature. We call it the wave constant-mean-curvature equation (or wave CMC for short). The equation will be for maps u which are defined on a domain in the $(2+1)$ -dimensional Minkowski space \mathbb{R}^{1+2} , on which we use coordinates (t, x, y) , and which maps into \mathbb{R}^3 . In fact, a map $u: [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is said to be a solution of the wave CMC, if on $[0, T] \times \mathbb{R}^2$ we have

$$(3) \quad (-\partial_t^2 + \Delta)u = 2u_x \wedge u_y$$

where Δ is the Laplacian on \mathbb{R}^2 acting componentwise on the three components of u , and $u_x \wedge u_y$ is the cross product of the two vectors u_x and u_y in \mathbb{R}^3 (hence the null form Q_{12} arises here). The stationary (elliptic) analog of this equation is

$$(4) \quad \Delta u = 2u_x \wedge u_y.$$

This is an interesting equation because if u solves $\Delta u = 2Hu_x \wedge u_y$ for some function H on \mathbb{R}^2 and satisfies the conformal conditions $|u_x| = |u_y| = 1$ and $u_x \cdot u_y = 0$ everywhere, then the image of u is a surface with mean curvature H in \mathbb{R}^3 . (4) is the special case of the equation $\Delta u = 2Hu_x \wedge u_y$ when $H \equiv 1$; hence the name wave CMC for (3). We recall that (4) has been studied by many authors in connection to semi-linear elliptic systems of partial differential equations; see e.g. Hildebrandt [7], Wente [31], [32], Brezis-Coron [1], [2], Struwe [23], [24], [25], Chanillo-Malchiodi [4] and Caldiroli-Musina [3]. In particular, bubbling phenomena for (4) was first studied by Brezis-Coron [2], and a more refined bubbling analysis was done in Chanillo-Malchiodi [4].

We also remind the reader that in Chanillo-Yung [5], Theorem 7, it is shown that (3) blows up in finite time if the initial energy exceeds that of the primary bubble of [2], the Riemann sphere with winding number one. Thus we are naturally led to understanding possible natures of the blow-up in [5].

Another motivation for us in studying (3) comes from the study of the energy-critical focusing semilinear wave equation. It is an equation for a scalar-valued function u on \mathbb{R}^{1+n} , $n \geq 3$, given by

$$(5) \quad (\partial_t^2 - \Delta)u = |u|^{4/(n-2)}u,$$

and it is also sometimes called the wave Yamabe equation, since it is the wave analog of the Yamabe equation

$$-\Delta u = |u|^{4/(n-2)}u$$

in conformal geometry. Kenig and Merle have developed in [8] a concentration compactness-rigidity approach to establish global well-posedness of (5) under a suitable class of initial data. A fundamental step in their work is the following result:

Theorem 1 (Kenig-Merle [8]). *Suppose $3 \leq n \leq 5$, and $(u_0, u_1) \in \dot{H}^1 \times L^2$ on \mathbb{R}^n is such that*

$$\int_{\mathbb{R}^n} \nabla u_0 u_1 = 0, \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}, \quad \text{and} \quad E(u_0, u_1) < E(W, 0),$$

where $W(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-\frac{n-2}{2}}$ is the ‘groundstate’ stationary solution to (5), and

$$E(u_0, u_1) := \int_{\mathbb{R}^n} \frac{1}{2} (|\nabla u_0|^2 + |u_1|^2) - \frac{n-2}{2n} |u|^{\frac{2n}{n-2}}$$

is the conserved energy of (5). Suppose also that $u: [-1, 0) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a solution to (5), with initial data $u|_{t=-1} = u_0$, $\partial_t u|_{t=-1} = u_1$. For $t \in [-1, 0)$, let

$$U_t(x) = (-t)^{n/2} (\nabla u)(t, -tx), \quad V_t(x) = (-t)^{n/2} (\partial_t u)(t, -tx).$$

If the solution $u(t, x)$ does not extend beyond $t = 0$, then the set

$$\{(U_t, V_t): t \in [-1, 0)\}$$

cannot have compact closure in $\dot{H}^1 \times L^2$.

Theorem 1 occupies a substantial portion of Section 6 of [8], leading to a unique continuation problem for a degenerate elliptic equation, Proposition 6.12 in [8]. In particular, this rules out the existence of self-similar solutions to (5).

Our future goal is to also apply concentration compactness-rigidity method to study equation (3). As a first step, we show in this paper that the wave CMC (3) does not admit self-similar blow-ups. More precisely, we prove in Section 2 the following result:

Theorem 2. *Suppose $v \in C^2(\mathbb{R}^2, \mathbb{R}^3)$ is such that*

$$(6) \quad u(x, y, t) = v\left(\frac{x}{t}, \frac{y}{t}\right)$$

is a solution to the wave CMC

$$(-\partial_t^2 + \Delta)u = 2u_x \wedge u_y \quad \text{on } \{t > 0\}.$$

Then v is constant on \mathbb{D} , where \mathbb{D} is the unit disc on \mathbb{R}^2 .

Note that by scaling and dimensional considerations, the self-similar blow-up can only be of the type (6).

The analogue of this statement for wave maps have been long known; see e.g. Chapter 7.5 of the monograph of Shatah-Struwe [20].

Our approach to Theorem 2 is inspired by that in [20]. Using a Pohozaev type identity from [4], we are also led to a unique continuation problem for a degenerate elliptic equation in self-similar coordinates. But since we are in dimension two, we may apply the uniformization theorem like in [20], and reduce matters to a unique continuation theorem of Hartman-Wintner [4], [6].

We are making the qualitative assumption $v \in C^2$ in the theorem, only to justify various integration by parts arguments in our proof.

We now turn our attention to local well-posedness of the initial value problem for the wave CMC (3). An easy scaling argument reveals that the wave CMC (3) is critical for initial data in $\dot{H}^1 \times L^2$. Well-posedness at such sharp regularity seems way out of reach at this point, because of the lack of sufficiently powerful Strichartz estimates in (2+1) dimensions. On the other hand, we take this chance to point out that the wave CMC (3) is locally well-posed, for initial data in $\dot{H}^{3/2} \times \dot{H}^{1/2}$; this is basically an observation that dates back to Klainerman-Machedon [9] (see Remark 3 in [9]).

Theorem 3. *Given any $K > 0$, there exists a small $T > 0$, and a constant $A > 0$ (both depending only on K) such that for any initial data $(u_0, u_1) \in \dot{H}^{3/2} \times \dot{H}^{1/2}$ with*

$$\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} \leq K,$$

the Cauchy problem

$$(-\partial_t^2 + \Delta)u = 2u_x \wedge u_y, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1$$

has a unique solution u on $[0, T]$ with

$$u \in C_{[0,T]}^0 \dot{H}^{3/2}, \quad \partial_t u \in C_{[0,T]}^0 \dot{H}^{1/2}, \quad \|u_x \wedge u_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq A,$$

in the sense that u solves the following integral equation for $t \in [0, T]$:

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} 2u_x \wedge u_y(s) ds.$$

Furthermore, the map

$$(u_0, u_1) \mapsto (u, \partial_t u)$$

$$\dot{H}^{3/2} \times \dot{H}^{1/2} \rightarrow C_{[0,T]}^0 \dot{H}^{3/2} \times C_{[0,T]}^0 \dot{H}^{1/2}$$

is continuous on the set $\{(u_0, u_1) : \|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} \leq K\}$.

Here $C_{[0,T]}^0 \dot{H}^s$ is the set of maps $u : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with

$$\|u\|_{C_{[0,T]}^0 \dot{H}^s} := \sup_{t \in [0,T]} \left(\int_{\mathbb{R}^2} |(-\Delta_{x,y})^{s/2} u(t, x, y)|^2 dx dy \right)^{1/2} < \infty,$$

and similarly one defines $L_{[0,T]}^p \dot{H}^s$. We provide a proof of Theorem 3 in Section 3, for the convenience of the reader.

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2. NON-EXISTENCE OF SELF-SIMILAR BLOW-UPS

Proof of Theorem 2. By finite speed of propagation, we may assume that v is defined only on $\overline{\mathbb{D}}$, and consider the solution u only in the light cone

$$\Gamma := \{(x, y, t) : \sqrt{x^2 + y^2} \leq t\}.$$

We will also forget about the values of v outside $\overline{\mathbb{D}}$.

Now we introduce self-similar variables

$$\tau = \sqrt{t^2 - x^2 - y^2}, \quad \rho = \frac{\sqrt{x^2 + y^2}}{t}, \quad \rho e^{i\theta} = \frac{x + iy}{t}$$

which is a re-parametrization of Γ . (τ is well-defined since we are now in the light cone Γ .) We also write $v = v(\rho, \theta)$ for $\rho \in [0, 1]$, $\theta \in [0, 2\pi]$. Then the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2$$

on Γ becomes

$$ds^2 = -d\tau^2 + \frac{\tau^2 d\rho^2}{(1 - \rho^2)^2} + \frac{\tau^2 \rho^2 d\theta^2}{1 - \rho^2}$$

in the new (τ, ρ, θ) coordinate system, i.e.

$$(7) \quad g_{\tau\tau} = -1, \quad g_{\rho\rho} = \frac{\tau^2}{(1 - \rho^2)^2}, \quad \text{and} \quad g_{\theta\theta} = \frac{\tau^2 \rho^2}{1 - \rho^2}.$$

In fact,

$$\tau^2 = t^2 - (x^2 + y^2) = t^2 - \rho^2 t^2 = (1 - \rho^2)t^2,$$

so

$$t = \frac{\tau}{\sqrt{1 - \rho^2}}.$$

Also, $x = \rho t \cos \theta$, $y = \rho t \sin \theta$, so

$$x = \frac{\tau \rho}{\sqrt{1 - \rho^2}} \cos \theta, \quad y = \frac{\tau \rho}{\sqrt{1 - \rho^2}} \sin \theta.$$

It follows that

$$\begin{aligned} dt &= \frac{1}{\sqrt{1-\rho^2}} d\tau + \frac{\tau\rho}{(1-\rho^2)^{3/2}} d\rho \\ dx &= \frac{\rho}{\sqrt{1-\rho^2}} \cos\theta d\tau + \frac{\tau}{(1-\rho^2)^{3/2}} \cos\theta d\rho - \frac{\tau\rho}{\sqrt{1-\rho^2}} \sin\theta d\theta \\ dy &= \frac{\rho}{\sqrt{1-\rho^2}} \sin\theta d\tau + \frac{\tau}{(1-\rho^2)^{3/2}} \sin\theta d\rho + \frac{\tau\rho}{\sqrt{1-\rho^2}} \cos\theta d\theta, \end{aligned}$$

and (7) follows. From (7), we have

$$\begin{aligned} \sqrt{|\det(g_{ij})|} &= \frac{\tau^2\rho}{(1-\rho^2)^{3/2}}, \\ g^{\tau\tau} &= -1, \quad g^{\rho\rho} = \frac{(1-\rho^2)^2}{\tau^2}, \quad \text{and} \quad g^{\theta\theta} = \frac{1-\rho^2}{\tau^2\rho^2}. \end{aligned}$$

It follows that the wave operator $-\partial_t^2 + \Delta_{x,y}$ becomes

$$\begin{aligned} &\frac{(1-\rho^2)^{3/2}}{\tau^2\rho} \left[\partial_\tau \left(-\frac{\tau^2\rho}{(1-\rho^2)^{3/2}} \partial_\tau \cdot \right) + \partial_\rho \left(\frac{\tau^2\rho}{(1-\rho^2)^{3/2}} \frac{(1-\rho^2)^2}{\tau^2} \partial_\rho \cdot \right) \right. \\ &\quad \left. + \partial_\theta \left(\frac{\tau^2\rho}{(1-\rho^2)^{3/2}} \frac{1-\rho^2}{\tau^2\rho^2} \partial_\theta \cdot \right) \right] \\ &= -\frac{(1-\rho^2)^{3/2}}{\tau^2\rho} \left[\partial_\tau \left(\frac{\tau^2\rho}{(1-\rho^2)^{3/2}} \partial_\tau \cdot \right) - \partial_\rho \left(\rho\sqrt{1-\rho^2} \partial_\rho \cdot \right) - \left(\frac{1}{\rho\sqrt{1-\rho^2}} \partial_\theta^2 \cdot \right) \right] \end{aligned}$$

Furthermore,

$$\begin{aligned} \partial_x &= \frac{\partial\tau}{\partial x} \partial_\tau + \frac{\partial\rho}{\partial x} \partial_\rho + \frac{\partial\theta}{\partial x} \partial_\theta \\ &= -\frac{x}{\sqrt{t^2-x^2-y^2}} \partial_\tau + \frac{x}{t\sqrt{x^2+y^2}} \partial_\rho - \frac{y}{x^2+y^2} \partial_\theta \\ &= -\frac{\rho \cos\theta}{\sqrt{1-\rho^2}} \partial_\tau + \frac{\sqrt{1-\rho^2} \cos\theta}{\tau} \partial_\rho - \frac{\sqrt{1-\rho^2} \sin\theta}{\tau\rho} \partial_\theta \end{aligned}$$

and similarly

$$\begin{aligned} \partial_y &= \frac{\partial\tau}{\partial y} \partial_\tau + \frac{\partial\rho}{\partial y} \partial_\rho + \frac{\partial\theta}{\partial y} \partial_\theta \\ &= -\frac{y}{\sqrt{t^2-x^2-y^2}} \partial_\tau + \frac{y}{t\sqrt{x^2+y^2}} \partial_\rho + \frac{x}{x^2+y^2} \partial_\theta \\ &= -\frac{\rho \sin\theta}{\sqrt{1-\rho^2}} \partial_\tau + \frac{\sqrt{1-\rho^2} \sin\theta}{\tau} \partial_\rho + \frac{\sqrt{1-\rho^2} \cos\theta}{\tau\rho} \partial_\theta. \end{aligned}$$

Applying these to $v(\rho, \theta)$, and noting that it is independent of τ , we see that the wave CMC for u becomes

$$\frac{(1 - \rho^2)^{3/2}}{\tau^2 \rho} \left[\partial_\rho \left(\rho \sqrt{1 - \rho^2} \partial_\rho v \right) + \frac{1}{\rho \sqrt{1 - \rho^2}} \partial_\theta^2 v \right] = 2 \frac{1 - \rho^2}{\tau^2 \rho} v_\rho \wedge v_\theta,$$

i.e.

$$(8) \quad \rho \sqrt{1 - \rho^2} \partial_\rho \left(\rho \sqrt{1 - \rho^2} \partial_\rho v \right) + \partial_\theta^2 v = 2 \rho v_\rho \wedge v_\theta.$$

Now take the dot product of both sides with v_ρ . Then the right hand side vanishes, and we get

$$\frac{1}{2} \partial_\rho \left| \rho \sqrt{1 - \rho^2} v_\rho \right|^2 + v_\rho \cdot \partial_\theta^2 v = 0.$$

Integrating this in θ , and integrating by parts in the last term, we get

$$\frac{d}{d\rho} \int_0^{2\pi} [\rho^2(1 - \rho^2)|v_\rho|^2 - |v_\theta|^2] d\theta = 0,$$

i.e.

$$(9) \quad \int_0^{2\pi} [\rho^2(1 - \rho^2)|v_\rho|^2 - |v_\theta|^2] d\theta$$

is a constant independent of ρ . Letting $\rho \rightarrow 0$, we see that this constant is zero (note that $v_\theta = O(\rho)$ as $\rho \rightarrow 0$, if v is differentiable at 0). So when $\rho = 1$, the integral (9) is equal to 0. It follows that

$$\int_0^{2\pi} |v_\theta|^2 d\theta = 0 \quad \text{when } \rho = 1,$$

i.e. v is a constant on $\partial\mathbb{D}$. Note that the wave CMC has no zeroth order term. Hence we may subtract a constant from v , to make

$$v = 0 \quad \text{on } \partial\mathbb{D},$$

and we will do so from now on.

Now introduce a new variable

$$\sigma = \exp \left(- \int_\rho^1 \frac{ds}{s \sqrt{1 - s^2}} \right) = \frac{\rho}{1 + \sqrt{1 - \rho^2}}$$

such that

$$\sigma \frac{\partial}{\partial \sigma} = \rho \sqrt{1 - \rho^2} \frac{\partial}{\partial \rho}.$$

(Note that as ρ varies between $[0, 1]$, σ also varies between $[0, 1]$.) Then equation (8) for v becomes

$$\sigma \partial_\sigma (\sigma \partial_\sigma v) + \partial_\theta^2 v = 2 \frac{\sigma}{\sqrt{1 - \rho^2}} v_\sigma \wedge v_\theta,$$

i.e.

$$(10) \quad v_{\sigma\sigma} + \frac{1}{\sigma}v_{\sigma} + \frac{1}{\sigma^2}v_{\theta\theta} = 2\frac{1}{\sigma\sqrt{1-\rho^2}}v_{\sigma} \wedge v_{\theta}.$$

Let $z = \sigma e^{i\theta}$. Then on the left hand side we have then the flat Laplacian on the z -plane. On the right hand side we have something normal to the surface parametrized by v . So we can apply the strategy of Chanillo-Malchiodi [4], Lemma 3.1. More precisely, we take the dot product of both sides of the equation with σv_{σ} . Then

$$\sigma v_{\sigma} \cdot \left[v_{\sigma\sigma} + \frac{1}{\sigma}v_{\sigma} + \frac{1}{\sigma^2}v_{\theta\theta} \right] = 0$$

on the unit disc \mathbb{D} in the z -plane. We integrate over \mathbb{D} using polar coordinates:

$$\begin{aligned} 0 &= \int_0^1 \int_0^{2\pi} \sigma v_{\sigma} \cdot \left[v_{\sigma\sigma} + \frac{1}{\sigma}v_{\sigma} + \frac{1}{\sigma^2}v_{\theta\theta} \right] \sigma d\theta d\sigma \\ &= \int_0^1 \int_0^{2\pi} \frac{\partial}{\partial \sigma} \left(\frac{1}{2}\sigma^2 |v_{\sigma}|^2 - \frac{1}{2}|v_{\theta}|^2 \right) + \frac{\partial}{\partial \theta} (v_{\sigma} \cdot v_{\theta}) d\theta d\sigma \\ &= \int_0^{2\pi} \left[\frac{1}{2}\sigma^2 |v_{\sigma}|^2 - \frac{1}{2}|v_{\theta}|^2 \right]_{\sigma=0}^{\sigma=1} d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{2}|v_{\sigma}|^2 \right]_{\sigma=1} d\theta \end{aligned}$$

(the last equality following from $v_{\theta} = 0$ when $\sigma = 1$ (note $\sigma = 1$ if and only if $\rho = 1$), and that

$$v_{\theta} = \rho(v_y \cos \theta - v_x \sin \theta) = O(\sigma |\nabla_{x,y} v|) \rightarrow 0$$

as $\sigma \rightarrow 0$). Hence

$$v_{\sigma} = 0 \quad \text{on } \{\sigma = 1\}.$$

Now we extend v so that $v = 0$ when $\sigma > 1$. Then from the above, $v(z) \in C^1(\mathbb{C} \setminus \{0\})$. We can then apply the unique continuation technique of Hartman-Wintner. More precisely, from Theorem 2 of [6], we have:

Theorem 4 (Hartman-Wintner [6]). *Suppose $r \in \mathbb{N}$, U is an open set in \mathbb{C} , and $v \in C^1(U, \mathbb{R}^r)$, and there exists continuous (matrix-valued) functions d, e, f on U such that*

$$(11) \quad \int_{\partial\Omega} g(z) \frac{\partial v}{\partial z} dz = \int_{\Omega} g(z) \left[d(z) \frac{\partial v}{\partial z} + e(z) \frac{\partial v}{\partial \bar{z}} + f(z) v(z) \right] dz \wedge d\bar{z}$$

for all piecewise smooth relatively compact domains Ω of U and all holomorphic functions g on $\overline{\Omega}$. If there exists a point $z_0 \in U$ such that

$$(12) \quad \lim_{z \rightarrow z_0} \frac{v(z)}{(z - z_0)^n} = 0 \quad \text{for all } n \in \mathbb{N},$$

then

$$v \equiv 0 \quad \text{on } U.$$

We verify that the conditions of the above theorem is met, when $r = 3$ and $U = \mathbb{C} \setminus \{0\}$: First, since $v \in C^1(\mathbb{C} \setminus \{0\})$, and is supported in $\overline{\mathbb{D}}$, we have, for any piecewise smooth relatively compact domain $\Omega \subset \mathbb{C} \setminus \{0\}$, that

$$\begin{aligned} \int_{\partial\Omega} g(z) \frac{\partial v}{\partial z} dz &= \int_{\partial(\Omega \cap \mathbb{D})} g(z) \frac{\partial v}{\partial z} dz \\ &= \int_{\Omega \cap \mathbb{D}} \frac{\partial}{\partial \bar{z}} \left(g(z) \frac{\partial v}{\partial z} \right) d\bar{z} \wedge dz \\ &= \int_{\Omega \cap \mathbb{D}} g(z) \frac{\partial^2 v}{\partial z \partial \bar{z}} d\bar{z} \wedge dz. \end{aligned}$$

But by (10), on \mathbb{D} we have

$$\frac{1}{4} \frac{\partial^2 v}{\partial z \partial \bar{z}} = v_{\sigma\sigma} + \frac{1}{\sigma} v_{\sigma} + \frac{1}{\sigma^2} v_{\theta\theta} = 2 \frac{1}{\sqrt{1-\rho^2}} v_{\sigma} \wedge \frac{1}{\sigma} v_{\theta},$$

and

$$\frac{1}{\sqrt{1-\rho^2}} v_{\sigma} = \frac{1}{\sqrt{1-\rho^2}} \frac{\rho \sqrt{1-\rho^2}}{\sigma} v_{\rho} = \frac{\rho}{\sigma} v_{\rho} = \frac{1}{1 + \sqrt{1-\rho^2}} v_{\rho}$$

is continuous up to $\{\sigma = 1\}$. Also, $\frac{1}{\sigma} v_{\theta}$ is a linear combination of $\frac{\partial v}{\partial z}$ and $\frac{\partial v}{\partial \bar{z}}$ on $\mathbb{C} \setminus \{0\}$. Hence one can find continuous (matrix-valued) functions d and e on $\overline{\mathbb{D}} \setminus \{0\}$, such that

$$\frac{\partial^2 v}{\partial z \partial \bar{z}} = d(z) \frac{\partial v}{\partial z} + e(z) \frac{\partial v}{\partial \bar{z}}$$

on $\mathbb{D} \setminus \{0\}$. Extending d and e continuously to \mathbb{C} , and using that v vanishes outside $\overline{\mathbb{D}}$, we see that (11) is satisfied with $f = 0$. Also, (12) is satisfied at any $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Hence Theorem 4 implies that $v \equiv 0$ on $\mathbb{C} \setminus \{0\}$, which also implies $v(0) = 0$ by continuity. In particular, we have $v(z) = 0$ for all $|z| < 1$, i.e. $v(\rho, \theta) = 0$ whenever $\rho < 1$, as desired.

□

3. LOCAL WELL-POSEDNESS IN $\dot{H}^{3/2}$

Proof of Theorem 3. Let $\square = (-\partial_t^2 + \Delta)$ be the D'Alembertian on \mathbb{R}^{1+2} . Recall the null form Q_{12} from (2). Note that the wave CMC (3) is a system of equations, that can be written in the components (u_1, u_2, u_3) of u as

$$\begin{aligned}\square u_1 &= 2Q_{12}(u_2, u_3) \\ \square u_2 &= 2Q_{12}(u_3, u_1) \\ \square u_3 &= 2Q_{12}(u_1, u_2).\end{aligned}$$

Also, as was observed in Klainerman-Machedon [9], we have the following estimates for the null form Q_{12} on \mathbb{R}^{1+2} : if $f, g \in \dot{H}^{1/2}(\mathbb{R}^2)$, and

$$\phi_{\pm} := \frac{e^{\pm it\sqrt{-\Delta}}}{\sqrt{-\Delta}} f, \quad \psi_{\pm} := \frac{e^{\pm it\sqrt{-\Delta}}}{\sqrt{-\Delta}} g,$$

then

$$(13) \quad \|(-\Delta)^{1/4} Q_{12}(\phi_+, \psi_+)\|_{L^2(\mathbb{R}^{1+2})} \leq C \|f\|_{\dot{H}^{1/2}} \|g\|_{\dot{H}^{1/2}}.$$

(We briefly recall the proof of this at the end of the section, for the convenience of the reader.) The same continues to hold, if the signs $(+, +)$ on the left hand side are replaced by any of the choices $(-, -)$, $(+, -)$ and $(-, +)$. Thus if $u, v: [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$, with

$$\begin{aligned}\square u &= F, & u|_{t=0} &= u_0, & \partial_t u|_{t=0} &= u_1, \\ \square v &= G, & v|_{t=0} &= v_0, & \partial_t v|_{t=0} &= v_1,\end{aligned}$$

then

$$(14) \quad \|u_x \wedge v_y\|_{L^2_{[0,T]} \dot{H}^{1/2}} \leq C (\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|F\|_{L^1_{[0,T]} \dot{H}^{1/2}}) \cdot (\|v_0\|_{\dot{H}^{3/2}} + \|v_1\|_{\dot{H}^{1/2}} + \|G\|_{L^1_{[0,T]} \dot{H}^{1/2}}).$$

Also, the standard energy estimate shows that

$$(15) \quad \|u\|_{C^0_{[0,T]} \dot{H}^{3/2}} + \|\partial_t u\|_{C^0_{[0,T]} \dot{H}^{1/2}} \leq 2(\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|F\|_{L^1_{[0,T]} \dot{H}^{1/2}}).$$

Now to prove the theorem, let K be given, and set

$$A = \max\{2K, 4CK^2\}$$

where from now on C is the constant in (14). Let $T > 0$ be sufficiently small, so that

$$(16) \quad 4T^{1/2} \leq \frac{1}{2},$$

$$(17) \quad 2T^{1/2} A \leq K,$$

and

$$(18) \quad 2CT^{1/2}(K + 2T^{1/2}A) \leq \frac{1}{4}.$$

To prove existence, we fix initial data $(u_0, u_1) \in \dot{H}^{3/2} \times \dot{H}^1$ with

$$\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} \leq K.$$

Let

$$u^{(0)} := 0,$$

and for $k \geq 0$, let $u^{(k+1)}$ solve

$$\square u^{(k+1)} = 2u_x^{(k)} \wedge u_y^{(k)}, \quad u^{(k+1)}|_{t=0} = u_0, \quad \partial_t u^{(k+1)}|_{t=0} = u_1.$$

We will prove, by induction, that for all $k \geq 0$,

$$(19) \quad \|u^{(k+1)} - u^{(k)}\|_{C_{[0,T]}^0 \dot{H}^{3/2}} + \|\partial_t u^{(k+1)} - \partial_t u^{(k)}\|_{C_{[0,T]}^0 \dot{H}^{1/2}} \leq \frac{A}{2^k}$$

$$(20) \quad \|u_x^{(k+1)} \wedge u_y^{(k+1)}\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq A$$

$$(21) \quad \|u_x^{(k+1)} \wedge u_y^{(k+1)} - u_x^{(k)} \wedge u_y^{(k)}\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq \frac{A}{2^k}.$$

In fact, first consider the case $k = 0$. Then from (15), we have

$$\|u^{(1)}\|_{C_{[0,T]}^0 \dot{H}^{3/2}} + \|\partial_t u^{(1)}\|_{C_{[0,T]}^0 \dot{H}^{1/2}} \leq 2(\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^1}) \leq 2K;$$

from (14), we have

$$\|u_x^{(1)} \wedge u_y^{(1)}\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq C(\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}})^2 \leq CK^2.$$

By our choice of A , this proves (19), (20) and (21) when $k = 0$.

Now suppose $k \geq 1$. Then by (15),

$$\begin{aligned} & \|u^{(k+1)} - u^{(k)}\|_{C_{[0,T]}^0 \dot{H}^{3/2}} + \|\partial_t u^{(k+1)} - \partial_t u^{(k)}\|_{C_{[0,T]}^0 \dot{H}^{1/2}} \\ & \leq 2\|2u_x^{(k)} \wedge u_y^{(k)} - 2u_x^{(k-1)} \wedge u_y^{(k-1)}\|_{L_{[0,T]}^1 \dot{H}^{1/2}} \\ & \leq 4T^{1/2} \frac{A}{2^{k-1}} \\ & \leq \frac{A}{2^k}. \end{aligned}$$

(The second-to-last inequality follows from (21) for $k - 1$ in place of k , and the last inequality from (16).) Also, by (14),

$$\begin{aligned} & \|u_x^{(k+1)} \wedge u_y^{(k+1)}\|_{L^2_{[0,T]} \dot{H}^{1/2}} \\ & \leq C(\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|2u_x^{(k)} \wedge u_y^{(k)}\|_{L^1_{[0,T]} \dot{H}^{1/2}})^2 \\ & \leq C(K + 2T^{1/2}\|u_x^{(k)} \wedge u_y^{(k)}\|_{L^2_{[0,T]} \dot{H}^{1/2}})^2 \\ & \leq C(K + 2T^{1/2}A)^2 \leq C(2K)^2 \leq A, \end{aligned}$$

(The second inequality follows from (20) with k replaced by $k - 1$, and the third inequality from (17).) Finally, note that

$$u_x^{(k+1)} \wedge u_y^{(k+1)} - u_x^{(k)} \wedge u_y^{(k)} = u_x^{(k+1)} \wedge (u^{(k+1)} - u^{(k)})_y + (u^{(k+1)} - u^{(k)})_x \wedge u_y^{(k)}$$

But by (14),

$$\begin{aligned} & \|u_x^{(k+1)} \wedge (u^{(k+1)} - u^{(k)})_y\|_{L^2_{[0,T]} \dot{H}^{1/2}} \\ & \leq C(\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|2u_x^{(k)} \wedge u_y^{(k)}\|_{L^1_{[0,T]} \dot{H}^{1/2}}) \|2u_x^{(k)} \wedge u_y^{(k)} - 2u_x^{(k-1)} \wedge u_y^{(k-1)}\|_{L^1_{[0,T]} \dot{H}^{1/2}} \\ & \leq 2CT^{1/2}(K + 2T^{1/2}A) \cdot \frac{A}{2^{k-1}} \\ & \leq \frac{A}{2^{k+1}}. \end{aligned}$$

(The second-to-last inequality follows from (20) and (21) with k replaced by $k - 1$, and the last inequality from (18).) Similarly, one can show

$$\|(u^{(k+1)} - u^{(k)})_x \wedge u_y^{(k)}\|_{L^2_{[0,T]} \dot{H}^{1/2}} \leq \frac{A}{2^{k+1}}.$$

Together they prove (21). This completes our proof of (19), (20) and (21).

Let X be the Banach space

$$\{u : u \in C^0_{[0,T]} \dot{H}^{3/2}, \partial_t u \in C^0_{[0,T]} \dot{H}^{1/2}\}$$

with the natural norm. Then by (19), $u^{(k)}$ is Cauchy in X . We write u for the limit of $u^{(k)}$ in X . Also, the sequence $u_x^{(k)} \wedge u_y^{(k)}$ is Cauchy in $L^2_{[0,T]} \dot{H}^{1/2}$, by (21). We write F for the limit of $u_x^{(k)} \wedge u_y^{(k)}$ in $L^2_{[0,T]} \dot{H}^{1/2}$. In particular,

$$(22) \quad \|F\|_{L^2_{[0,T]} \dot{H}^{1/2}} \leq A,$$

and $u_x^{(k)} \wedge u_y^{(k)}$ also converges to F in $L^1_{[0,T]} \dot{H}^{1/2}$. Now for $t \in [0, T]$,

(23)

$$u^{(k+1)}(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} 2u_x^{(k)} \wedge u_y^{(k)}(s) ds.$$

Passing to limit in X , we then see that

(24)

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} 2F(s) ds.$$

(The convergence of the right hand side in X is guaranteed by the energy estimate (15), and the convergence of $u_x^{(k)} \wedge u_y^{(k)}$ to F in $L^1_{[0,T]} \dot{H}^{1/2}$.)

On the other hand, we claim that $u_x^{(k)} \wedge u_y^{(k)}$ converges to $u_x \wedge u_y$ in $L^2_{[0,T]} \dot{H}^{1/2}$. Assuming the claim for the moment, we then see that $u_x \wedge u_y = F$ on $[0, T] \times \mathbb{R}^2$, so (24) becomes

$$u(t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} 2u_x \wedge u_y(s) ds,$$

and (22) implies

$$\|u_x \wedge u_y\|_{L^2_{[0,T]} \dot{H}^{1/2}} \leq A,$$

as desired. So we now move on to prove the claim.

To do so, note that

$$u_x^{(k)} \wedge u_y^{(k)} - u_x \wedge u_y = u_x^{(k)} \wedge (u^{(k)} - u)_y + (u^{(k)} - u)_x \wedge u_y,$$

so by (14) and (24),

$$\begin{aligned} & \|u_x^{(k)} \wedge u_y^{(k)} - u_x \wedge u_y\|_{L^2_{[0,T]} \dot{H}^{1/2}} \\ & \leq C(\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|2u_x^{(k-1)} \wedge u_y^{(k-1)}\|_{L^1_{[0,T]} \dot{H}^{1/2}}) \|2u_x^{(k-1)} \wedge u_y^{(k-1)} - 2F\|_{L^1_{[0,T]} \dot{H}^{1/2}} \\ & \quad + C\|2u_x^{(k-1)} \wedge u_y^{(k-1)} - 2F\|_{L^1_{[0,T]} \dot{H}^{1/2}} (\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|2F\|_{L^1_{[0,T]} \dot{H}^{1/2}}) \\ & \leq 2C(K + 2T^{1/2}A)T^{1/2} \|2u_x^{(k-1)} \wedge u_y^{(k-1)} - 2F\|_{L^2_{[0,T]} \dot{H}^{1/2}} \\ & \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. (The second inequality follows from (20) and (22), and the last convergence follows from our definition of F .) This proves our claim, and hence our existence result.

Next, for uniqueness, assume that $u, v: [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$ solves

$$\square u = 2u_x \wedge u_y, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1$$

$$\square v = 2v_x \wedge v_y, \quad v|_{t=0} = u_0, \quad \partial_t v|_{t=0} = u_1$$

with

$$u, v \in C_{[0,T]}^0 \dot{H}^{3/2}, \quad \partial_t u, \partial_t v \in C_{[0,T]}^0 \dot{H}^{1/2},$$

and

$$\|u_x \wedge u_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq A, \quad \|v_x \wedge v_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq A.$$

We will show

$$u = v \quad \text{on } [0, T].$$

To do so, note that

$$\square(u - v) = 2u_x \wedge u_y - 2v_x \wedge v_y, \quad (u - v)|_{t=0} = 0, \quad \partial_t(u - v)|_{t=0} = 0.$$

So by the energy estimate (15),

$$\|u - v\|_{C_{[0,T]}^0 \dot{H}^{3/2}} + \|\partial_t(u - v)\|_{C_{[0,T]}^0 \dot{H}^{1/2}} \leq CT^{1/2} \|2u_x \wedge u_y - 2v_x \wedge v_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}}.$$

Now by (14),

$$\begin{aligned} & \|2u_x \wedge u_y - 2v_x \wedge v_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \\ & \leq 2(\|u_x \wedge (u - v)_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} + \|(u - v)_x \wedge v_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}}) \\ & \leq 2C(\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|2u_x \wedge u_y\|_{L_{[0,T]}^1 \dot{H}^{1/2}}) \|2u_x \wedge u_y - 2v_x \wedge v_y\|_{L_{[0,T]}^1 \dot{H}^{1/2}} \\ & \quad + 2C\|2u_x \wedge u_y - 2v_x \wedge v_y\|_{L_{[0,T]}^1 \dot{H}^{1/2}} (\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|2v_x \wedge v_y\|_{L_{[0,T]}^1 \dot{H}^{1/2}}) \\ & \leq 4CT^{1/2}(K + 2T^{1/2}A) \|2u_x \wedge u_y - 2v_x \wedge v_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \\ & \leq \frac{1}{2} \|2u_x \wedge u_y - 2v_x \wedge v_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \end{aligned}$$

by (18). Thus

$$\|2u_x \wedge u_y - 2v_x \wedge v_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} = 0,$$

which implies

$$\|u - v\|_{C_{[0,T]}^0 \dot{H}^{3/2}} + \|\partial_t(u - v)\|_{C_{[0,T]}^0 \dot{H}^{1/2}} = 0.$$

So $u = v$ on $[0, T]$, as desired.

Finally, we prove the continuous dependence of the solution on initial data. Suppose (u_0, u_1) and (v_0, v_1) are initial data, so that

$$\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} \leq K, \quad \|v_0\|_{\dot{H}^{3/2}} + \|v_1\|_{\dot{H}^{1/2}} \leq K,$$

and

$$\|u_0 - v_0\|_{\dot{H}^{3/2}} + \|u_1 - v_1\|_{\dot{H}^{1/2}} \leq \varepsilon.$$

Let u, v be the unique solution to

$$\begin{aligned} \square u &= 2u_x \wedge u_y, & u|_{t=0} &= u_0, & \partial_t u|_{t=0} &= u_1 \\ \square v &= 2v_x \wedge v_y, & v|_{t=0} &= v_0, & \partial_t v|_{t=0} &= v_1 \end{aligned}$$

with

$$u, v \in C_{[0,T]}^0 \dot{H}^{3/2}, \quad \partial_t u, \partial_t v \in C_{[0,T]}^0 \dot{H}^{1/2},$$

and

$$\|u_x \wedge u_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq A, \quad \|v_x \wedge v_y\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq A.$$

We claim

$$(25) \quad \|u - v\|_{C_{[0,T]}^0 \dot{H}^{3/2}} + \|\partial_t u - \partial_t v\|_{C_{[0,T]}^0 \dot{H}^{1/2}} \leq B\varepsilon$$

where

$$B = \max\{4, 4C(K + 2T^{1/2}A)\}.$$

This will prove continuous dependence on initial data.

To see this, recall that u, v are the limits in our space X of a sequence $u^{(k)}, v^{(k)}$ respectively, where $u^{(0)} = v^{(0)} = 0$, and

$$\square u^{(k+1)} = 2u_x^{(k)} \wedge u_y^{(k)}, \quad u^{(k+1)}|_{t=0} = u_0, \quad \partial_t u^{(k+1)}|_{t=0} = u_1,$$

$$\square v^{(k+1)} = 2v_x^{(k)} \wedge v_y^{(k)}, \quad v^{(k+1)}|_{t=0} = v_0, \quad \partial_t v^{(k+1)}|_{t=0} = v_1$$

for all $k \geq 0$. We will prove, by induction, that

$$(26) \quad \|u_x^{(k)} \wedge u_y^{(k)} - v_x^{(k)} \wedge v_y^{(k)}\|_{L_{[0,T]}^2 \dot{H}^{1/2}} \leq B\varepsilon$$

for all $k \geq 0$. Assuming this for the moment. Then the energy estimate (15) shows that for all $k \geq 0$,

$$\begin{aligned} & \|u^{(k+1)} - v^{(k+1)}\|_{C_{[0,T]}^0 \dot{H}^{3/2}} + \|\partial_t u^{(k+1)} - \partial_t v^{(k+1)}\|_{C_{[0,T]}^0 \dot{H}^{1/2}} \\ & \leq 2(\|u_0 - v_0\|_{\dot{H}^{3/2}} + \|u_1 - v_1\|_{\dot{H}^{1/2}} + \|2u_x^{(k)} \wedge u_y^{(k)} - 2v_x^{(k)} \wedge v_y^{(k)}\|_{L_{[0,T]}^1 \dot{H}^{1/2}}) \\ & \leq 2(\varepsilon + 2T^{1/2}B\varepsilon) \\ & \leq 2\varepsilon + \frac{B}{2}\varepsilon \\ & \leq B\varepsilon. \end{aligned}$$

(The third-to-last inequality follows from (26), and the second-to-last follows from (16). The last inequality follows from our choice of B that $B \geq 4$.) Letting $k \rightarrow \infty$, (25) follows.

Thus it remains to prove (26). It clearly holds when $k = 0$. Now suppose $k \geq 1$. By (14),

$$\begin{aligned}
& \|u_x^{(k)} \wedge u_y^{(k)} - v_x^{(k)} \wedge v_y^{(k)}\|_{L^2_{[0,T]} \dot{H}^{1/2}} \\
& \leq \|u_x^{(k)} \wedge (u^{(k)} - v^{(k)})_y\|_{L^2_{[0,T]} \dot{H}^{1/2}} + \|(u^{(k)} - v^{(k)})_x \wedge v_y^{(k)}\|_{L^2_{[0,T]} \dot{H}^{1/2}} \\
& \leq C(\|u_0\|_{\dot{H}^{3/2}} + \|u_1\|_{\dot{H}^{1/2}} + \|2u_x^{(k-1)} \wedge u_y^{(k-1)}\|_{L^1_{[0,T]} \dot{H}^{1/2}}) \\
& \quad \cdot (\|u_0 - v_0\|_{\dot{H}^{3/2}} + \|u_1 - v_1\|_{\dot{H}^{1/2}} + \|2u_x^{(k-1)} \wedge u_y^{(k-1)} - 2v_x^{(k-1)} \wedge v_y^{(k-1)}\|_{L^1_{[0,T]} \dot{H}^{1/2}}) \\
& \quad + C(\|u_0 - v_0\|_{\dot{H}^{3/2}} + \|u_1 - v_1\|_{\dot{H}^{1/2}} + \|2u_x^{(k-1)} \wedge u_y^{(k-1)} - 2v_x^{(k-1)} \wedge v_y^{(k-1)}\|_{L^1_{[0,T]} \dot{H}^{1/2}}) \\
& \quad \cdot (\|v_0\|_{\dot{H}^{3/2}} + \|v_1\|_{\dot{H}^{1/2}} + \|2v_x^{(k-1)} \wedge v_y^{(k-1)}\|_{L^1_{[0,T]} \dot{H}^{1/2}})
\end{aligned}$$

By our choice of initial data (u_0, u_1) and (v_0, v_1) , and using (20) with our induction hypothesis (26) with k replaced by $k-1$, this is bounded by

$$2C(K + 2T^{1/2}A)(\varepsilon + 2T^{1/2}B\varepsilon),$$

which by choice of B (so that $2C(K + 2T^{1/2}A) \leq \frac{B}{2}$) and (18) is bounded by

$$\frac{B}{2}\varepsilon + \frac{B}{2}\varepsilon = B\varepsilon.$$

This completes our induction, and hence the proof of our theorem. \square

We briefly outline the proof of (13). The space-time Fourier transform of $\phi_+ \cdot \psi_+$ is the convolution of $\tilde{\phi}_+$ with $\tilde{\psi}_+$. We compute this convolution by testing it against a test function $\varphi(\xi, \tau)$:

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{R}^2} (\tilde{\phi}_+ * \tilde{\psi}_+)(\xi, \tau) \varphi(\xi, \tau) d\xi d\tau \\
& = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{\phi}_+(\xi, \tau) \tilde{\psi}_+(\xi', \tau') \varphi(\xi + \xi', \tau + \tau') d\xi d\xi' d\tau d\tau' \\
& = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\hat{f}(\xi)}{|\xi|} \frac{\hat{g}(\xi')}{|\xi'|} \varphi(\xi + \xi', |\xi| + |\xi'|) d\xi d\xi' \\
& = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\hat{f}(\xi - \xi')}{|\xi - \xi'|} \frac{\hat{g}(\xi')}{|\xi'|} \varphi(\xi, |\xi - \xi'| + |\xi'|) d\xi d\xi'.
\end{aligned}$$

Now write ξ' in polar coordinates: $\xi' = \rho\omega$ where $\rho > 0$ and $\omega \in \mathbb{S}^1$. Then the above integral becomes

$$(27) \quad \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} \int_0^\infty \frac{\hat{f}(\xi - \rho\omega)}{|\xi - \rho\omega|} \hat{g}(\rho\omega) \varphi(\xi, |\xi - \rho\omega| + \rho) d\rho d\omega d\xi.$$

We change variables from ρ to τ , where τ is a new variable defined as

$$\tau = |\xi - \rho\omega| + \rho;$$

then

$$\begin{aligned}\tau^2 &= |\xi - \rho\omega|^2 + \rho^2 + 2\rho|\xi - \rho\omega| \\ &= |\xi|^2 - 2\rho\xi \cdot \omega + \rho^2 + \rho^2 + 2\rho(\tau - \rho) \\ &= |\xi|^2 + 2\rho(\tau - \xi \cdot \omega),\end{aligned}$$

which implies

$$(28) \quad \rho = \rho(\xi, \tau, \omega) = \frac{\tau^2 - |\xi|^2}{2(\tau - \xi \cdot \omega)}.$$

(Incidentally, this shows the change of variables is legitimate; it is a (smooth) bijection of $\rho \in [0, \infty)$, to $\tau \in [|\xi|, \infty)$.) Hence (27) becomes

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{S}^1} \chi_{\tau > |\xi|} \frac{\hat{f}(\xi - \rho\omega)}{|\xi - \rho\omega|} \hat{g}(\rho\omega) \frac{\partial \rho}{\partial \tau} \varphi(\xi, \tau) d\omega d\tau d\xi,$$

(henceforth ρ will be defined by ξ, τ, ω as in (28)). This shows the convolution $\tilde{\phi}_+ * \tilde{\psi}_+$ is given by

$$(29) \quad \tilde{\phi}_+ * \tilde{\psi}_+(\xi, \tau) = \chi_{\tau > |\xi|} \int_{\mathbb{S}^1} \frac{\hat{f}(\xi - \rho\omega)}{|\xi - \rho\omega|} \hat{g}(\rho\omega) \frac{\partial \rho}{\partial \tau} d\omega.$$

We further simplify this formula:

$$|\xi - \rho\omega| = \tau - \rho = \tau - \frac{\tau^2 - |\xi|^2}{2(\tau - \xi \cdot \omega)} = \frac{\tau^2 - 2\tau\xi \cdot \omega + |\xi|^2}{2(\tau - \xi \cdot \omega)}.$$

Also,

$$(30) \quad \frac{\partial \rho}{\partial \tau} = \frac{(\tau - \xi \cdot \omega)(2\tau) - (\tau^2 - |\xi|^2)}{2(\tau - \xi \cdot \omega)^2} = \frac{\tau^2 - 2\tau\xi \cdot \omega + |\xi|^2}{2(\tau - \xi \cdot \omega)^2}.$$

Hence

$$\frac{1}{|\xi - \rho \cdot \omega|} \frac{\partial \rho}{\partial \tau} = \frac{1}{\tau - \xi \cdot \omega} = \frac{2\rho}{\tau^2 - |\xi|^2},$$

and (29) becomes

$$(31) \quad \tilde{\phi}_+ * \tilde{\psi}_+(\xi, \tau) = \frac{2\chi_{\tau > |\xi|}}{\tau^2 - |\xi|^2} \int_{\mathbb{S}^1} \hat{f}(\xi - \rho\omega) \hat{g}(\rho\omega) \rho d\omega.$$

Now to prove (13), note that

$$|\mathcal{F}((-\Delta)^{1/4} Q_{12}(\phi_+, \psi_+))(\xi, \tau)| = |\xi|^{1/2} |\mathcal{F}(Q_{12}(\phi_+, \psi_+))(\xi, \tau)|$$

and (at least formally)

$$\begin{aligned} & \mathcal{F}(Q_{12}(\phi_+, \psi_+))(\xi, \tau) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} (\xi_1 \xi'_2 - \xi_2 \xi'_1) \tilde{\phi}_+(\xi - \xi', \tau - \tau') \tilde{\psi}_+(\xi', \tau') d\xi' d\tau'. \end{aligned}$$

So putting absolute values inside the integral, and using

$$|\xi|^{1/2} \leq |\xi - \xi'|^{1/2} + |\xi'|^{1/2},$$

we see that one has

$$\begin{aligned} & |\mathcal{F}((-\Delta)^{1/4} Q_{12}(\phi_+, \psi_+))(\xi, \tau)| \\ & \leq \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\xi_1 \xi'_2 - \xi_2 \xi'_1| |\xi - \xi'|^{1/2} |\tilde{\phi}_+(\xi - \xi', \tau - \tau')| |\tilde{\psi}_+(\xi', \tau')| d\xi' d\tau' \\ & \quad + \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\xi_1 \xi'_2 - \xi_2 \xi'_1| |\tilde{\phi}_+(\xi - \xi', \tau - \tau')| |\xi'|^{1/2} |\tilde{\psi}_+(\xi', \tau')| d\xi' d\tau' \\ & = I + II \end{aligned}$$

The integral II can be brought, via a change of variables $\xi' \mapsto \xi - \xi'$, $\tau' \mapsto \tau - \tau'$, into

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |\xi_1 \xi'_2 - \xi_2 \xi'_1| |\xi - \xi'|^{1/2} |\tilde{\psi}_+(\xi - \xi', \tau - \tau')| |\tilde{\phi}_+(\xi', \tau')| d\xi' d\tau'$$

which is the same as integral I , except now the roles of ϕ_+ and ψ_+ (hence the roles of f and g) are reversed. Since the right hand side of our desired estimate (13) is symmetric in f and g , it suffices now to bound the integral I . But I can be computed by testing against a test function as above. We then get, in a similar manner that we derived (31), that

$$I(\xi, \tau) = \frac{2\chi_{\tau > |\xi|}}{\tau^2 - |\xi|^2} \int_{\mathbb{S}^1} \rho |\xi_1 \omega_2 - \xi_2 \omega_1| |\hat{F}(\xi - \rho\omega)| |\hat{g}(\rho\omega)| \rho d\omega$$

where

$$F := (-\Delta)^{1/4} f.$$

It follows that

$$|I(\xi, \tau)|^2 \lesssim \frac{\chi_{\tau > |\xi|}}{(\tau^2 - |\xi|^2)^2} \int_{\mathbb{S}^1} \rho^2 |\xi_1 \omega_2 - \xi_2 \omega_1|^2 |\hat{F}(\xi - \rho\omega)|^2 |\hat{g}(\rho\omega)|^2 \rho^2 d\omega.$$

We now integrate with respect to ξ and τ , use Fubini's theorem, and change variables $\tau \mapsto \rho$. Then

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \int_{\mathbb{R}} |I(\xi, \tau)|^2 d\tau d\xi \\
 (32) \quad & \lesssim \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} \int_{\rho=0}^{\rho=\infty} \frac{\rho^2 |\xi_1 \omega_2 - \xi_2 \omega_1|^2}{(\tau^2 - |\xi|^2)^2} \frac{\partial \tau}{\partial \rho} |\hat{F}(\xi - \rho\omega)|^2 |\hat{g}(\rho\omega)|^2 \rho^2 d\rho d\omega d\xi.
 \end{aligned}$$

But we claim

$$(33) \quad \frac{\rho^2 (\xi_1 \omega_2 - \xi_2 \omega_1)^2}{(\tau^2 - |\xi|^2)^2} \frac{\partial \tau}{\partial \rho} \leq \frac{1}{2}.$$

In fact, remembering $\frac{\partial \tau}{\partial \rho}$ is the reciprocal of $\frac{\partial \rho}{\partial \tau}$, which we computed in (30), we have

$$\begin{aligned}
 \frac{\rho^2 (\xi_1 \omega_2 - \xi_2 \omega_1)^2}{(\tau^2 - |\xi|^2)^2} \frac{\partial \tau}{\partial \rho} &= \frac{\rho^2 (\xi_1 \omega_2 - \xi_2 \omega_1)^2 [2(\tau - \xi \cdot \omega)^2]}{(\tau^2 - |\xi|^2)^2 (\tau^2 - 2\tau \xi \cdot \omega + |\xi|^2)} \\
 &= \frac{(\xi_1 \omega_2 - \xi_2 \omega_1)^2}{2(\tau^2 - 2\tau \xi \cdot \omega + |\xi|^2)}
 \end{aligned}$$

But the denominator can be simplified, by writing

$$\begin{aligned}
 \tau^2 - 2\tau \xi \cdot \omega + |\xi|^2 &= (\tau - \xi \cdot \omega)^2 + |\xi|^2 - (\xi \cdot \omega)^2 \\
 &= (\tau - \xi \cdot \omega)^2 + (\xi_1 \omega_2 - \xi_2 \omega_1)^2
 \end{aligned}$$

(the last equality holds since we are in 2-dimensions). Thus we see that the above quotient is bounded by $1/2$, as was claimed in (33).

Now by (32) and (33), we see that

$$\begin{aligned}
 \int_{\mathbb{R}^2} \int_{\mathbb{R}} |I(\xi, \tau)|^2 d\tau d\xi &\leq C \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} \int_{\rho=0}^{\rho=\infty} |\hat{F}(\xi - \rho\omega)|^2 |\hat{g}(\rho\omega)|^2 \rho^2 d\rho d\omega d\xi \\
 &= C \|F\|_{L^2}^2 \|g\|_{\dot{H}^{1/2}}^2 \\
 &= C \|f\|_{\dot{H}^{1/2}}^2 \|g\|_{\dot{H}^{1/2}}^2.
 \end{aligned}$$

This completes the proof of (13).

REFERENCES

- [1] Haïm Brezis and Jean-Michel Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Comm. Pure Appl. Math. **37** (1984), no. 2, 149–187.
- [2] ———, *Convergence of solutions of H-systems or how to blow bubbles*, Arch. Rational Mech. Anal. **89** (1985), no. 1, 21–56.
- [3] Paolo Caldiroli and Roberta Musina, *The Dirichlet problem for H-systems with small boundary data: blow-up phenomena and nonexistence results*, Arch. Rational Mech. Anal. **181** (2006), no. 1, 1–42.

- [4] Sagun Chanillo and Andrea Malchiodi, *Asymptotic Morse theory for the equation $\Delta v = 2v_x \wedge v_y$* , Communications in Analysis and Geometry **13** (2005), no. 1, 187–251.
- [5] Sagun Chanillo and Po-Lam Yung, *Wave equations associated to Liouville systems and constant mean curvature equations*, Adv. Math. **235** (2013), 187–207.
- [6] Philip Hartman and Aurel Wintner, *On the local behavior of solutions of non-parabolic partial differential equations*, Amer. J. Math. **75** (1953), 449–476.
- [7] Stefan Hildebrandt, *On the Plateau problem for surfaces of constant mean curvature*, Comm. Pure Appl. Math. **23** (1970), 97–114.
- [8] Carlos E. Kenig and Frank Merle, *Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation*, Acta Math. **201** (2008), no. 2, 147–212.
- [9] Sergiu Klainerman and Matei Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), no. 9, 1221–1268.
- [10] ———, *Smoothing estimates for null forms and applications*, Duke Math. J. **81** (1995), no. 1, 99–133 (1996). A celebration of John F. Nash, Jr.
- [11] ———, *On the regularity properties of a model problem related to wave maps*, Duke Math. J. **87** (1997), no. 3, 553–589.
- [12] Sergiu Klainerman and Sigmund Selberg, *Remark on the optimal regularity for equations of wave maps type*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 901–918.
- [13] ———, *Bilinear estimates and applications to nonlinear wave equations*, Commun. Contemp. Math. **4** (2002), no. 2, 223–295.
- [14] Joachim Krieger, *Global regularity of wave maps from \mathbf{R}^{3+1} to surfaces*, Comm. Math. Phys. **238** (2003), no. 1-2, 333–366.
- [15] ———, *Global regularity of wave maps from \mathbf{R}^{2+1} to H^2 . Small energy*, Comm. Math. Phys. **250** (2004), no. 3, 507–580.
- [16] Joachim Krieger and Wilhelm Schlag, *Concentration compactness for critical wave maps*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2012.
- [17] J. Krieger, W. Schlag, and D. Tataru, *Renormalization and blow up for charge one equivariant critical wave maps*, Invent. Math. **171** (2008), no. 3, 543–615.
- [18] Pierre Raphaël and Igor Rodnianski, *Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems*, Publ. Math. Inst. Hautes Études Sci. (2012), 1–122.
- [19] Igor Rodnianski and Jacob Sterbenz, *On the formation of singularities in the critical $O(3)$ σ -model*, Ann. of Math. (2) **172** (2010), no. 1, 187–242.
- [20] Jalal Shatah and Michael Struwe, *Geometric wave equations*, Courant Lecture Notes in Mathematics, vol. 2, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1998.
- [21] Jacob Sterbenz and Daniel Tataru, *Regularity of wave-maps in dimension $2+1$* , Comm. Math. Phys. **298** (2010), no. 1, 231–264.
- [22] ———, *Energy dispersed large data wave maps in $2+1$ dimensions*, Comm. Math. Phys. **298** (2010), no. 1, 139–230.
- [23] Michael Struwe, *Nonuniqueness in the Plateau problem for surfaces of constant mean curvature*, Arch. Rational Mech. Anal. **93** (1986), no. 2, 135–157.

- [24] ———, *Large H -surfaces via the mountain-pass-lemma*, Math. Ann. **270** (1985), no. 3, 441–459.
- [25] ———, *The existence of surfaces of constant mean curvature with free boundaries*, Acta Math. **160** (1988), no. 1-2, 19–64.
- [26] Terence Tao, *Global regularity of wave maps. I. Small critical Sobolev norm in high dimension*, Internat. Math. Res. Notices **6** (2001), 299–328.
- [27] ———, *Global regularity of wave maps. II. Small energy in two dimensions*, Comm. Math. Phys. **224** (2001), no. 2, 443–544.
- [28] Daniel Tataru, *Local and global results for wave maps. I*, Comm. Partial Differential Equations **23** (1998), no. 9-10, 1781–1793.
- [29] ———, *On global existence and scattering for the wave maps equation*, Amer. J. Math. **123** (2001), no. 1, 37–77.
- [30] ———, *Rough solutions for the wave maps equation*, Amer. J. Math. **127** (2005), no. 2, 293–377.
- [31] Henry C. Wente, *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. **26** (1969), 318–344.
- [32] ———, *The differential equation $\Delta x = 2H(x_u \wedge x_v)$ with vanishing boundary values*, Proc. Amer. Math. Soc. **50** (1975), 131–137.

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